§ The result

<u>Thm</u> There is a bijection

$$\left\{ \text{ECs}/\mathbb{Q} \text{ up to isogeny} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{l} \text{Normalized newforms of} \\ \cdot)\ \text{weight } 2 \\ \cdot)\ \text{level } \Gamma_0(N),\ \text{some } N \geq 1 \\ \cdot)\ \text{rational Fourier coeff.} \end{array} \right\}$$

It is given by $E \longleftrightarrow f$ if for almost all p,

$$\mathrm{tr}\left(F_p \in \mathrm{End}\left(\widetilde{E}_p\right)\right) = p\text{-th Fourier coeff } a_p \text{ of } f$$

@

absolute Frob = relative Frobenius since $\widetilde{E}_p / \mathbb{F}_p$

Special fibre $\mathbb{F}_p \underset{\mathbb{Z}}{\otimes} \widetilde{E}$ of integral model $\widetilde{E} / \mathbb{Z}[B^{-1}]$ $B =$ primes of bad red.

Here $\mathrm{tr}(F_p) = F_p + F_p^* \in \mathbb{Z}$.

@ RHS: f certain function on upper half-plane $\mathbb{H}$,

has expression $f(\tau) = \sum_{n \geq 1} a_n \cdot \exp(2\pi i \cdot n \cdot \tau)$

a_n called n-th <u>Fourier coefficient</u>

§ Idea of proof

Consider cohomology of modular curve:

étale cohom $H^1(\bar{\mathbb{Q}} \otimes_{\mathbb{Q}} X_K, \bar{\mathbb{Q}}_\ell)$

comparison isom $\longrightarrow \;\wr\!\wr$

Betti cohom $H^1(X_K(\mathbb{C}), \mathbb{C})$ — "usual" cohom. from Topology 1.

·) Modularity Thm $\Rightarrow$ Each E "occurs" in étale cohom

·) Hodge theory / $\mathbb{C}$ $\Rightarrow$ Betti cohom "=" space of modular forms of weight 2, level K

Then consider a huge ring(*), the Hecke algebra $\mathcal{H}$.
It acts naturally (compatibly w/ comparison iso) on both cohom groups, decomposing them into eigenspaces.
Matching eigenspaces by eigenvalues gives bijection from theorem.

(*) $\mathcal{H}$ still very simple, polynomial ring $\mathbb{Q}[T_p, \langle p\rangle^{\pm 1} : p \nmid K]$.

§ Modularity Thm

<u>Thm</u> (Wiles, Breuil - Conrad - Diamond - Taylor)

For every EC $E/\mathbb{Q}$, there is an $N \geq 1$ and a non-constant map $X_{\Gamma_0(N)} \xrightarrow{f} E$.

f called <u>modular parametrization</u> of E

<u>Notation</u> M_N; $M_{N,p}$ / $\mathbb{Z}[\frac{1}{N}]$ usual moduli spaces

Y_N ; $Y_{N,p}$:= $\mathbb{Q} \underset{\mathbb{Z}}{\otimes} M_N$ resp. $\mathbb{Q} \underset{\mathbb{Z}}{\otimes} M_{N,p}$

(smooth affine curves over Spec $\mathbb{Q}$.)

X_N, $X_{N,p}$:= unique smooth compactifications of Y_N resp. $Y_{N,p}$

(Two constructions: 1) Pick $Y \longrightarrow \mathbb{P}^1_{\mathbb{Q}}$ non-constant, put X := normalization of $\mathbb{P}^1_{\mathbb{Q}}$ in Y.

2) $|X| := \{$ valuations v on $\mathbb{Q}(Y)$ s.th. $v|_{\mathbb{Q}}$ trivial $\}/_\sim$

topology: closed sets = $\{v_1, \ldots, v_r\}$ v_i non-constant, $r < \infty$.

$\mathcal{O}_X(U) = \{ f \in \mathbb{Q}(Y) \mid v(f) \geq 0 \ \forall \ v \in U \}$.)

We use as black box that $\mathcal{M}_N$, $\mathcal{M}_{N,p}$ have natural smooth compactifications $\mathcal{M}_N^*$, $\mathcal{M}_{N,p}^*$ / $\mathbb{Z}[\frac{1}{N}]$.

Let $K \subseteq GL_2(\hat{\mathbb{Z}})$ any level.

Assume $K \cap GL_2(\mathbb{Z}_p) = GL_2(\mathbb{Z}_p)$

Pick N s.th. $K(N) \subseteq K$ and $(p, N) = 1$.

Define $\mathcal{M}_K$, $\mathcal{M}_{K,p}$, Y_K, etc. as quotient of $\mathcal{M}_N$, $\mathcal{M}_{N,p}$, Y_N, ... by $K/K(N)$.

$\Longrightarrow$ This defines $X_{\Gamma_0(N)} = X_{K_0(N)}$ from Thm.

($X_{\Gamma_0(N)}$ = coarse moduli of $(E, C \subseteq E)$, $C \cong \mathbb{Z}/N$ étale locally.)

<u>Reinterpretation in terms of the Jacobian.</u>

<u>Def</u> J_K := Jacobian of X_K

:= abelian variety representing $\mathrm{Pic}^\circ_{X_K/\mathbb{Q}}$.

$\mathrm{Pic}^\circ_{X_K/\mathbb{Q}}(S) = \{ \mathcal{L}$ on $S \times_{\mathrm{Spec}\,\mathbb{Q}} X_K$ s.th.

$\deg \mathcal{L}(s)|_C = 0 \quad \forall\, s \in S,\ C \in \pi_0(\kappa(s) \otimes X_K) \} / \mathrm{Pic}(S)$

(The defn. captures that X_K might be non-connected.

Note that $X_{\Gamma_0(N)}$ is geometrically connected since

$$\det : \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in GL_2(\mathbb{Z}/N) \right\} \longrightarrow (\mathbb{Z}/N)^\times$$

is surjective. This is the case of interest,

J is then usual Jacobian.)

<u>Prop (Poincaré reducibility)</u> k any field

$\{ AV/k$ w/ $\mathbb{Q} \otimes_{\mathbb{Z}} Hom(-,-) \}$ (isogeny category of AVs)

is semi-simple.

<u>Idea</u> Given: $Y \hookrightarrow X$ sub-AV. Pick

ample l.b. $\mathcal{L}$ on X, consider induced polarization

$\phi_{\mathcal{L}} : X \longrightarrow X^\vee$. Put $Z = \ker(X^\vee \longrightarrow Y^\vee)$.

Then one may show that $Y \times \phi_{\mathcal{L}}^{-1}(Z) \longrightarrow X$

is an isogeny. (cf. AV Lect 25 or Mumford §19 Thm 1) $\square$

$\Rightarrow \quad J_K \underset{\substack{\uparrow \\ \exists \text{ isogeny}}}{\sim} \prod_{i=1}^{r} A_i^{n_i}$

for uniquely determined $n_i \geq 1$ and <u>simple</u> abelian varieties with $\mathrm{Hom}(A_i, A_j) = 0$ $i \neq j$.

A <u>simple</u> $\underset{\text{def}}{=}$ $Z \subseteq A$ sub-AV $\Rightarrow$ $Z = 0$ or A

$\Leftrightarrow$ $\mathrm{End}^0(A)$ is a division algebra.

<u>Rmk</u> There always exist finite morphisms $X \to J$:
E.g. assume X connected, let $D \in \mathrm{Div}(X)$ any s.th.
$\deg D \neq 0$. Then $x \mapsto \mathcal{O}(D - \deg(D) \cdot [x])$
defines such a map. In this case, the composition

$$X^g \longrightarrow J^g \xrightarrow{\Sigma} J \qquad g = \text{genus of } X$$

is surjective & generically finite.

It follows that, for an AV A:

$\exists$ non-constant $X \longrightarrow A$ $\Leftrightarrow$ $\exists$ $J \xrightarrow{\neq 0} A$

$\Leftrightarrow$ $A \sim A_i$ for some i.

Example ECs are simple since of dimension 1.

There is a moduli space $\mathcal{A}_g/\mathbb{Q}$ of principally polarized AVs of dim g. It has $\dim \mathcal{A}_g = \frac{g(g+1)}{2}$.

In particular,

$$\dim \mathcal{A}_h \times \mathcal{A}_{g-h} < \dim \mathcal{A}_g \quad \forall\, 0<h<g.$$

$\Rightarrow$ "Most" AVs$/\mathbb{C}$ are not products of smaller-dim AVs up to isogeny, meaning they are simple.

Reformulation of Modularity Thm:

Every EC $E/\mathbb{Q}$ occurs in some $J_{\Gamma_0(N)}$.

§ Correspondences

<u>Why bother?</u>

G group, $G \curvearrowright X$, $K \subseteq G$ subgroup.

Then no G-action on $K\backslash X$ anymore!

Namely $Kx_1 = Kx_2$ does <u>not</u> imply

$$Kgx_1 = Kgx_2 \quad \text{in general.}$$

Only the set of <u>all possible choices</u> is well-defined:

$$g \bullet x := K\backslash KgKx \subseteq K\backslash X.$$

i.e. view $g \bullet -$ as multivalued map (= correspondence)

$$C_g := \{ (Kx_1, Kx_2) \mid \exists\, k_1, k_2 \text{ s.th. } gk_1x_1 = k_2x_2 \}$$

$$\Leftrightarrow \; k_2^{-1}gk_1 \cdot x_1 = x_2$$

Now let V = functions on X

(or sections of a G-equivariant bundle)

Then $G \curvearrowright V$ by translation and

K-invariants V^K = functions on $K\backslash X$

But V^K not G-stable in general, only

$$g \cdot v \in V^{gKg^{-1}} \text{ if } v \in V^K.$$

Assume however that $K \backslash KgK$ is finite, so

$$p_1 : C_g \longrightarrow K\backslash X \quad \text{is finite.}$$

Then can define

take fiber-wise sum

$$C_g : V^K \longrightarrow V^K, \quad f \longmapsto p_{1,*}\left(p_2|_{C_2}^* f\right)$$

$$\text{resp. } v \longmapsto \sum_{i \in I} h_i \cdot v \qquad \underline{K\text{-averaging}}$$

where $KgK = \coprod_{i \in I} K \cdot h_i$. ($I$ finite by assumpt.)

Our <u>case of interest</u>: $G = Gl_2$

$$G(\mathbb{Q}) \backslash \left(\mathbb{H}^{\pm} \times G(\mathbb{A}_f)\right) \circlearrowleft G(\mathbb{A}_f)$$

$\cup|$

K level subgroup.

Def X smooth proper curve $/ k$

$\mathrm{Corr}(X) :=$ $\mathbb{Q}$-vsp generated by

closed integral $Z \subseteq X \times X$ s.th.

$p_1, p_2 : Z \to X$ are finite loc free

(As X smooth, this is equivalent to $\dim Z = 1$

and $p_1, p_2|_Z$ non-constant.)

Becomes a ring ($\mathbb{Q}$-algebra) under

$$Z_1 * Z_2 := \mathrm{Im}\left(Z_1 \underset{p_2, X, p_1}{\times} Z_2 \xrightarrow{(p_1, p_2)} X \times X \right)$$

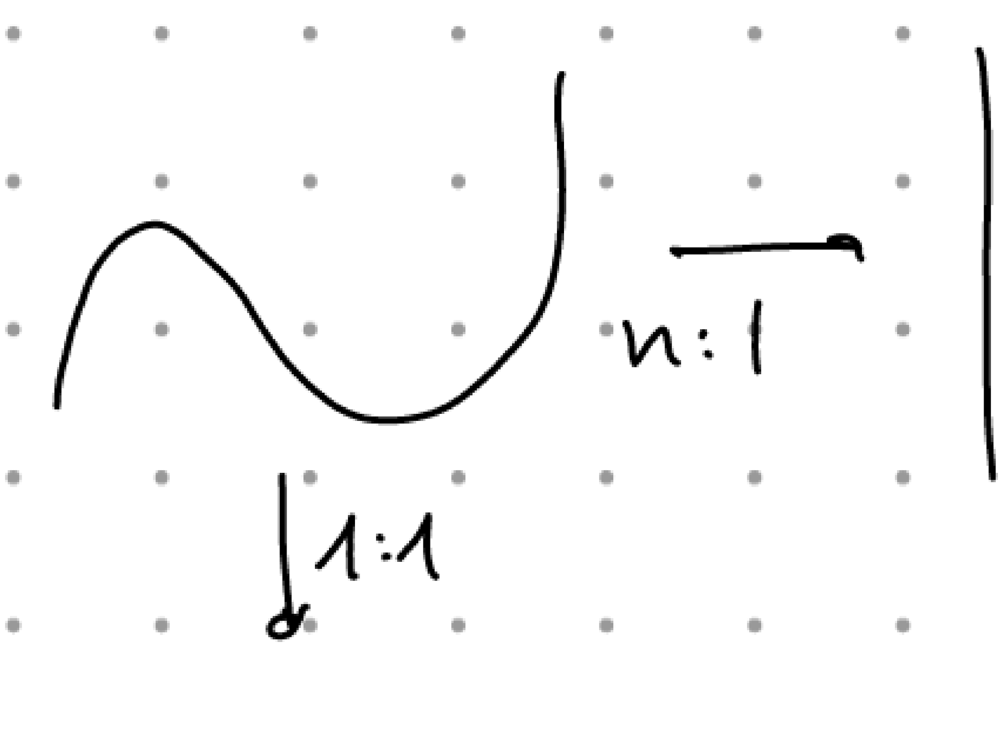
In following sense: $Z_1 \times Z_2 = \bigcup_{i=1}^{r} Y_i$ irred components.

Then RHS $= \sum [k(Y_i) : k(p(Y_i))] \cdot p(Y_i)$

$p = (p_1, p_2)$

<u>Example</u> $f : X \to X$ non-constant has graph

Γ_f



<u>Check</u> $\Gamma_f * \Gamma_g = \Gamma_{g \circ f}$

So correspondences generalize concept of maps!

There is a ring morphism $\operatorname{Corr}(X) \longrightarrow \operatorname{End}^\circ(J)$:

$$\underbrace{Z * \mathcal{L}}_{\text{line bundle on } S \times_k X} := \det\left(p_{1,*}\left(p_2^* \mathcal{L}\right)\right) \otimes \det\left(p_{1,*}\mathcal{O}_Z\right)^{-1}$$

(Here one has to put $\det 0 := \mathcal{O}_X$ to cover $Z = 0$.)

Concretely for $S = \operatorname{Spec} k$:

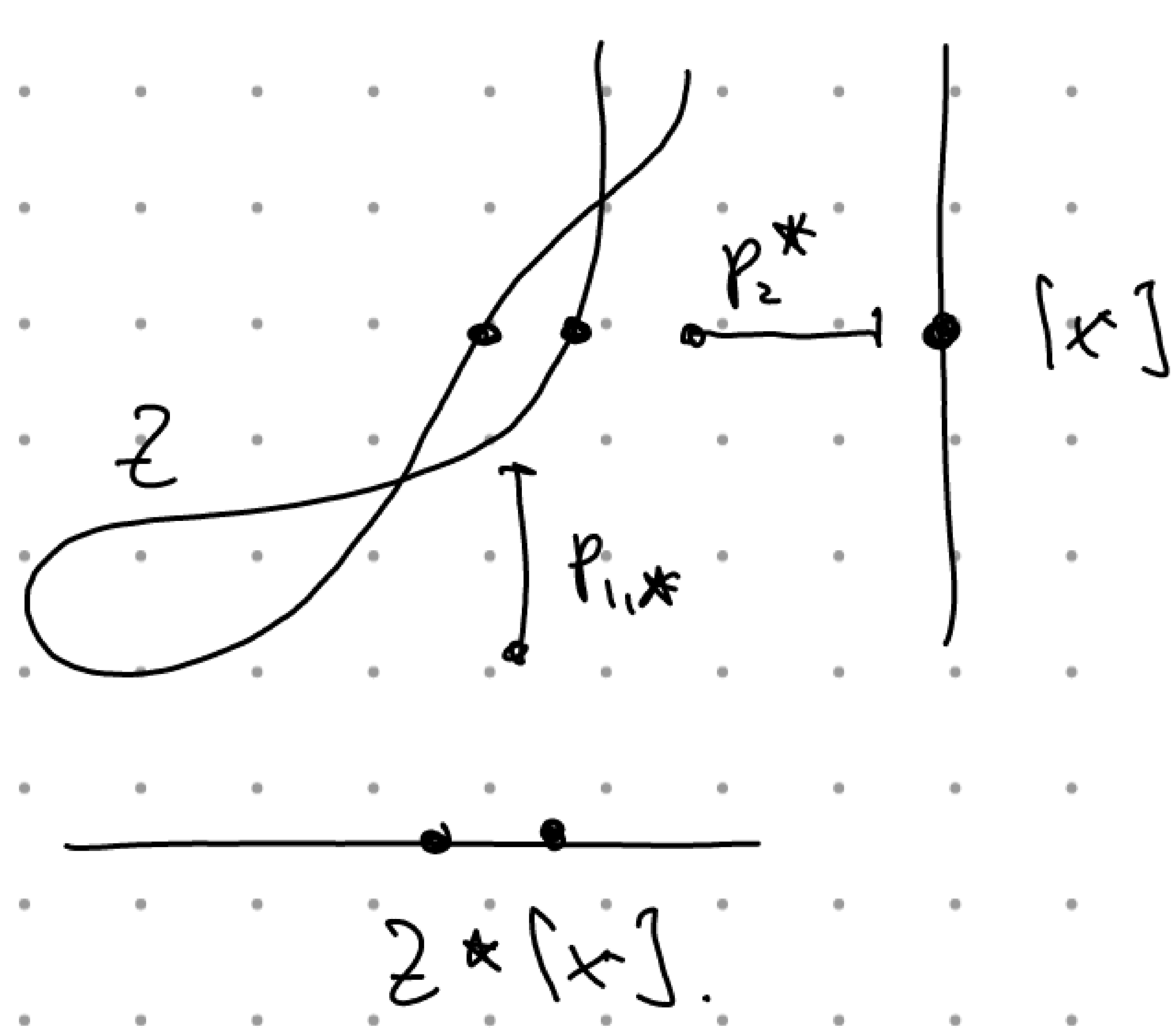
Consider $\mathcal{L} = \mathcal{O}_X([x])$ w/ $p_2^{-1}(x) \subseteq$ normal locus of Z.

Then $p_2^* \mathcal{L} = \mathcal{O}_Z\left(\sum_{p_2(z)=x} e_{p_2,z} \cdot [z]\right)$

and $v_{x'}\left(\det p_{1,*}\mathcal{O}_Z \longrightarrow \det p_{1,*} p_2^* \mathcal{L}\right)$

$$= \sum_{\substack{p_1(z)=x' \\ p_2(z)=x}} e_{p_1,z} \cdot f_{p_1,z} \cdot e_{p_2,z} \cdot [x']$$

$\Rightarrow$ $Z*-$ comes from the multi-valued map defined by Z.



From divisor description we see that

$$\deg \mathcal{L} = 0 \implies \deg Z * \mathcal{L} = 0,$$

so get map $J \longrightarrow J$ as claimed.

(Exercise: Check from divisor description that

$$Z_1 * (Z_2 * \mathcal{L}) = (Z_1 * Z_2) * \mathcal{L}.\ )$$

See Fulton §16 for more details.

§ Hecke correspondence

Let p prime, $K = GL_2(\mathbb{Z}_p) \subseteq GL_2(\mathbb{Q}_p)$

Def Hecke algebra $\mathcal{H}_p \underset{def}{=} \mathbb{Q}\left[GL_2(\mathbb{Z}_p) \backslash GL_2(\mathbb{Q}_p) / GL_2(\mathbb{Z}_p)\right]$

(finite linear combinations of double cosets.)

View as left and right K-invariant functions with compact support on $GL_2(\mathbb{Q}_p)$. Then multiplication is convolution

$$(f * g)(x) = \int_{GL_2(\mathbb{Q}_p)} f(y)\, g(y^{-1}x)\, dy$$

w.r.t. unique translation invariant measure s.t. $\mu(K) = 1$.

Elementary divisor thm:

$$K \backslash GL_2(\mathbb{Q}_p) / K = \left\{ K \cdot \begin{pmatrix} p^a & \\ & p^b \end{pmatrix} K \;\middle|\; a \geq b \right\}$$

$\implies$ $\mathbb{1}_{K\begin{pmatrix} p^a & \\ & p^b \end{pmatrix}K}$ form $\mathbb{Q}$-basis for $\mathcal{H}_p$

$\begin{pmatrix} p & \\ & p \end{pmatrix}$ lies in center, so $K\begin{pmatrix} p & \\ & p \end{pmatrix}K$

$= K\begin{pmatrix} p & \\ & p \end{pmatrix} = \begin{pmatrix} p & \\ & p \end{pmatrix}K$

Put $\langle p \rangle := \mathbb{1}_{K\begin{pmatrix} p & \\ & p \end{pmatrix}}$. Then

$$\langle p \rangle * \mathbb{1}_{K\begin{pmatrix} p^a & \\ & p^b \end{pmatrix}K} = \mathbb{1}_{K\begin{pmatrix} p^{a+1} & \\ & p^{b+1} \end{pmatrix}K}.$$

What about $T_{p^i} := \mathbb{1}_{K\begin{pmatrix} p^i & \\ & 1 \end{pmatrix}K}$?

Have following description: $GL_2(\mathbb{Q}_p) / GL_2(\mathbb{Z}_p) \cong \{ \underbrace{\Lambda \subseteq \mathbb{Q}_p^2}_{\text{lattices}} \}$

So $\mathcal{H}_p = \mathrm{Map}_c(\{\Lambda\}, \mathbb{Q})^{GL_2(\mathbb{Z}_p)}$

finite support

invariant under $GL_2(\mathbb{Z}_p)$-action on $\{\Lambda\}$.

Multivalued map $T_{p^i}(\Lambda) = \{ \Lambda' \subseteq \Lambda,\ \Lambda/\Lambda' \cong \mathbb{Z}/p^i \}$

$\leftrightarrow$ elementary divisor $(p^i, 1)$.

(Rmk $\# T_{p^i}(\Lambda) = \# \mathbb{P}^1(\mathbb{Z}/p^i) = p^{i-1}(p+1)$)

Then $T_{p^i} \curvearrowright \mathrm{Map}_c(\{\Lambda\}, \mathbb{Q})$ via

$$T_{p^i}\Big(\sum_r \lambda_r \cdot \mathbb{1}_{\Lambda_r}\Big) = \sum_r \lambda_r \cdot \mathbb{1}_{T_{p^i}(\Lambda_r)}$$

and this describes multiplication in $\mathcal{H}_p$!

$\multimap$ $T_p * T_p = T_{p^2} + (p+1)\langle p\rangle$

$$= \sum_{\Lambda' \subsetneq_1 \Lambda \subsetneq_1 \mathbb{Z}_p^2} \mathbb{1}_{\Lambda'}$$

For $\Lambda' = p \cdot \mathbb{Z}_p^2$, there are $p+1$ ways to squeeze $\Lambda' \subsetneq_1 \Lambda \subsetneq_1 \mathbb{Z}_p^2$

If $\mathbb{Z}_p^2/\Lambda' \cong \mathbb{Z}/p^2$, there is a unique squeezed $\Lambda' \subsetneq_1 \Lambda \subsetneq_1 \mathbb{Z}_p^2$.

Such arguments show:

Prop $\mathcal{H}_p = \mathbb{Q}[\langle p\rangle^{\pm 1}, T_p]$

($\cong \mathbb{Q}[S^{\pm 1}, T]$ commutative polynomial ring !)

Now back to global setting, $N \geq 1$. One obtains

$$\bigotimes_{p \nmid N} \mathcal{H}_p \longrightarrow \mathrm{Corr}(X_N)$$

•) $\langle p \rangle \longmapsto \langle p \rangle$ operator

$\langle p \rangle (E, \alpha) = (E, p\alpha)$ on Y_K

View as correspondence via graph $\Gamma_{\langle p \rangle}$.

•) $T_p \longmapsto X_{N,p}$

View $X_{N,p}$ as correspondence via the two maps

$$(E, \alpha, C) \longmapsto \big((E, \alpha), (E/C, \alpha \bmod C)\big)$$

(The multivalued map is thus $[E] \longmapsto \sum_{\substack{E \to E' \\ \text{of deg } p}} [E']$.)

<u>Upshot</u> Let $K(N) \subseteq K \subseteq GL_2(\hat{\mathbb{Z}})$ level, e.g. $K_0(N)$.

Obtain $\mathbb{Q}[\langle p \rangle^{\pm 1}, T_p \,;\, p \nmid N] \longrightarrow \mathrm{End}^0(J_K)$

$$\mathcal{H}^N := \bigotimes_{p \nmid N} \mathcal{H}_p \quad (= \mathbb{Q}[\langle p \rangle^{\pm 1}, T_p \,;\, p \nmid N])$$

<u>Hecke algebra away from N</u>

Recall May write $J_K \sim \prod_{i\in I} A_i^{n_i}$

in unique way with A_i simple, $A_i \not\sim A_j$ $i \neq j$.

Then $\mathrm{End}^0(J_K) = \prod_{i\in I} M_{n_i}(D_i)$ $D_i = \mathrm{End}^0(A_i)$
division $\mathbb{Q}$-algebra.

Contains the projections $e_i = (0, \dots, 0, 1_i, 0, \dots, 0)$

Thm 1) The image of $\mathcal{H}^N \longrightarrow \mathrm{End}^0(J_K)$

contains the idempotents e_i.

2) Given an EC $E/\mathbb{Q}$, there exists a choice of

level K s.th. $J_K \sim E \times A$

with $\mathrm{Hom}(E, A) = 0$. (Newform theory)
+ modularity thm.

Rmk This theorem is proved analytically by studying

modular forms, i.e. $H^1_{\mathrm{Betti}}(X_K(\mathbb{C}), \mathbb{C})$,

and then carries over through the comparison

iso with étale cohomology.

§ Eichler - Shimura

K s.th. $J_K \sim E \times A$, $\mathrm{Hom}(E,A) = 0$.

<u>Question</u> What is the composition

$$\mathcal{H}^N \longrightarrow \mathrm{End}^\circ(J_K) \xrightarrow{pr} \mathrm{End}^\circ(E)$$
$$T_p \longmapsto \quad ?$$

(E occurs for $K_0(N)$ and any subgroup $C \subseteq E$ is stable under $p\cdot$, so one knows a priori that $\langle p \rangle \mapsto \mathrm{id}_E$. The $\langle p \rangle$-operator only plays a role when refining the level, e.g. $\Gamma_1(N) \leftarrow K_1(N)$.)

<u>Thm</u> $T_p \mapsto a_p \in \mathbb{Z} \subseteq \mathrm{End}^\circ(E)$,

$$a_p = \mathrm{tr}(F_p) = F_p + F_p^* = F_p + V_p = p + 1 - \#\tilde{E}(\mathbb{F}_p)$$

(This is the corrected sign)

<u>Proof</u> :) $M_K / \mathbb{Z}[\frac{1}{N}]$ has a smooth compactification M_K^*

(Means $M_K^* \longrightarrow \mathrm{Spec}\, \mathbb{Z}[\frac{1}{N}]$ proper smooth, contains M_K as dense open.)

$\Longrightarrow$ J_K extends to abelian scheme over $\mathbb{Z}[\frac{1}{N}]$.

namely $\tilde{J}_K := \mathrm{Pic}^0_{M_K^*/\mathbb{Z}[\frac{1}{N}]}$.

·) Weil extension Thm:

$$\mathrm{End}(J_K) = \mathrm{End}(\tilde{J}_K)$$

In partic, the idempotent $\left(E \times A \xrightarrow{pr} E \overset{inc}{\hookrightarrow} E \times A\right)$ lifts to $\tilde{J}_K$, giving $\tilde{J}_K \sim \tilde{E} \times \tilde{A}$

In partic, E also has good reduction, model $\tilde{E}$.

·) Furthermore, (a compactification of) $M_{K,p}$ provides an integral model of T_p that describes the image of $T_{p,\mathbb{Q}}$ under Weil extn Thm

$$\mathrm{End}(J_K) \xrightarrow{\cong} \mathrm{End}(\tilde{J}_K).$$

& $$\mathrm{End}(E) \xrightarrow{\cong} \mathrm{End}(\tilde{E}).$$

·) By rigidity, $\mathrm{End}(\tilde{J}_K) \hookrightarrow \mathrm{End}(\mathbb{F}_\ell \otimes \tilde{J}_K)$

$$\mathrm{End}(\tilde{E}) \longrightarrow \mathrm{End}(\mathbb{F}_\ell \otimes \tilde{E})$$

for any prime $\ell \nmid N$.

We choose to work at the "bad" (= interesting) prime $\ell = p$.

·) $\mathbb{F}_p \otimes \tilde{J}_K$ = Jacobian of special fib $\overline{M_K^*}$

& interest lies with its endomorphism defined by special fibe of correspondence $\overline{M_{K,p}^*}$

(This just expresses functoriality properties of the Jacobian construction and the map

$$\mathrm{Corr}(X) \longrightarrow \mathrm{End}(J).)$$

·) In $\mathrm{Corr}(\overline{M_K^*})$, $\qquad F \qquad\qquad V$

$$\overline{M_{K,p}^*} = \Phi(\overline{M_K^*}) \sqcup w\Phi(\overline{M_K^*})$$

Identifying them with $\overline{M_K^*}$ via Φ and $w\Phi$, there are

$$\Phi: \begin{tikzcd} \overline{M_K^*} \arrow[r, "\mathrm{Frob}"] \arrow[d, "\mathrm{id}"] & \overline{M_K^*} \\ \overline{M_K^*} & \end{tikzcd} \qquad w\Phi: \begin{tikzcd} \overline{M_K^*} \arrow[r, "\langle p \rangle"] \arrow[d, "\mathrm{Frob}"] & \overline{M_K^*} \\ \overline{M_K^*} & \end{tikzcd}$$

The composition $F \circ V$ in $\mathrm{Corr}(\overline{M}_K^*)$ acts on $\widetilde{J}_p$ (in terms of divisors) as

$$D \longmapsto \mathrm{Frob}^* \mathrm{Frob}_* \langle p \rangle^* D = p \cdot \langle p \rangle^* D.$$

(Given $f: X \to Y$ degree d map of curves,

$$f_* f^* D = d \cdot D \;)$$

F itself acts as Verschiebung of $\widetilde{J}_p$ by functoriality.

$\Longrightarrow V = \langle p \rangle \cdot F^*$ dual isogeny.

·) Mentioned above: $\langle p \rangle \longmapsto 1$ in $\mathrm{End}(E)$

$$\Longrightarrow T_p \mapsto F + F^* = \mathrm{tr}(F) \in \mathrm{End}(E)$$

∎

On other hand $\exists$ 1-dimensional $\mathcal{H}^N$-stable summand

$$\mathbb{C} \cdot f_E \subseteq \{ \text{Modular forms of wt } 2, \text{ level } K \}$$

that corresponds to E under comparison iso étale $\rightsquigarrow$ Betti.

Thus $T_p \circlearrowright \mathbb{C} \cdot f_E$ also via a_p!

If f_E is normalized, then this reflects in its Fourier coeff. being a_p.